

Speed of sound of a spin balanced Fermi gas with s- and d-wave pairings across the BCS-BEC evolution

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The authors of a recent paper (PRA **87**, 013613 (2013)) argued that in fermionic systems with d-wave pairing the speed of sound is nonanalytic across the BCS-BEC crossover at the point where the chemical potential vanishes, regardless of the specific details of the interaction potential. On the contrary, the numerical results reported here suggest that the speed of sound across the BCS-BEC evolution of atomic Fermi gases with s- and d-wave pairings in two-dimensional square lattices is a smooth analytic function at the vanishing chemical potential.

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I. INTRODUCTION

It is widely accepted that the s-wave superfluidity of Fermi gases in two-dimensional optical lattices can be described at low energies and temperatures by a BCS theory with an effective on-site attractive interaction between square-lattice atoms that leads to a s-wave gap Δ_s . In the mean-field approximation, the number- and gap-equations for a fixed filling factor f at zero temperature are

$$1 = \frac{U}{N} \sum_{\mathbf{k}} \frac{1}{2\sqrt{\xi^2(\mathbf{k}) + \Delta_s^2}}, \quad (1)$$

$$f = \frac{1}{N} \sum_{\mathbf{k}} \left(1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi^2(\mathbf{k}) + \Delta_s^2}} \right), \quad (2)$$

where $\xi(\mathbf{k}) = 2J(1 - \cos k_x) + 2t(1 - \cos k_y) - \mu$ is the tight-binding dispersion energy, J is the tunneling strength of the atoms between nearest-neighbor sites of the two-dimensional (2D) lattice, μ is the chemical potential, and U is the strength of the attractive interaction (the lattice spacing is assumed to be $a = 1$).

It is evident from many numerical calculations, based on the Gaussian approximation,¹ the density-response-function approach,² or the Bethe-Salpeter (BS) formalism,³ that in fermion systems with s-wave pairing we have a smooth BCS-BEC crossover. For instance, the collective-mode dispersion within the Gaussian approximation at zero temperature is defined by the zeros of the following 2×2 secular determinant:

$$\begin{vmatrix} \frac{2}{U} + I_{\gamma,\gamma}^{0,0} + I_{l,l}^{0,0} - 2J_{\gamma,l}^{0,0} & I_{l,l}^{0,0} - I_{\gamma,\gamma}^{0,0} \\ I_{l,l}^{0,0} - I_{\gamma,\gamma}^{0,0} & \frac{2}{U} + I_{\gamma,\gamma}^{0,0} + I_{l,l}^{0,0} + 2J_{\gamma,l}^{0,0} \end{vmatrix}, \quad (3)$$

where $I_{\gamma,\gamma}^{0,0}$, $I_{l,l}^{0,0}$ and $J_{\gamma,l}^{0,0}$ are defined in Section II.

Years ago, it was pointed out that the Gaussian approximation can be obtained by summing diagrams corresponding only to the direct interaction.⁴ Diagrammatically, this means that the Gaussian approximation includes only contributions from the ladder diagrams. The BS formalism, as well as the density-response-function

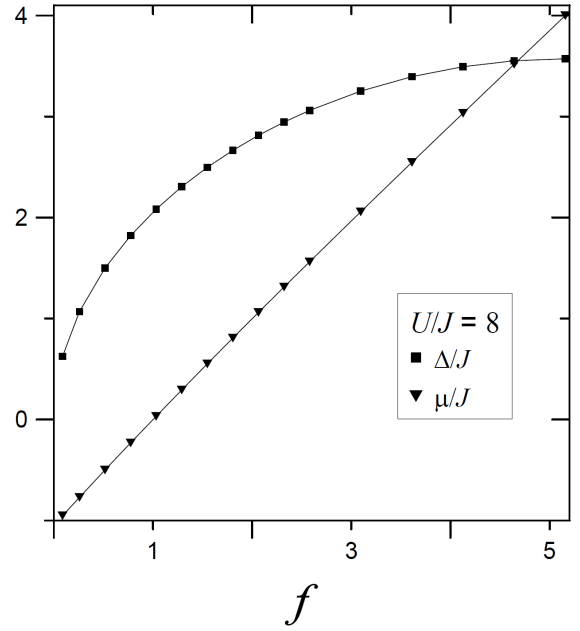


FIG. 1: The s-wave pairing gap Δ_s , and the chemical potential μ as functions of filling factor f (solid lines are guides to the eyes). The interaction strength is $U = 8J$, and the filling factor has been normalized such that $f = 1$ at $\mu = 0$.

approach, allow us to take into account contributions from both the ladder diagrams (direct interaction) and bubble diagrams (exchange interaction). When both interactions are taken into account, the secular determinant becomes:

$$\begin{vmatrix} 1 + UI_{\gamma,\gamma}^{0,0} & UI_{\gamma,l}^{0,0} & UI_{\gamma,\tilde{\gamma}}^{0,0} & UI_{\gamma,m}^{0,0} \\ UI_{\gamma,l}^{0,0} & 1 + UI_{l,l}^{0,0} & UI_{l,\tilde{\gamma}}^{0,0} & UI_{l,m}^{0,0} \\ UI_{\gamma,\tilde{\gamma}}^{0,0} & UI_{l,\tilde{\gamma}}^{0,0} & -1 + UI_{\tilde{\gamma},\tilde{\gamma}}^{0,0} & UI_{\tilde{\gamma},m}^{0,0} \\ UI_{\gamma,m}^{0,0} & UI_{l,m}^{0,0} & UI_{\tilde{\gamma},m}^{0,0} & 1 + UI_{m,m}^{0,0} \end{vmatrix} \quad (4)$$

In Fig. 1, we have shown the solutions of the number- and gap-equations across the BSC-BEC evolution as functions of the filling factor f . The interaction strength is $U = 8J$. The chemical potential vanishes at $f = 0.1938$, and we have normalized the filling factor such that $f = 1$

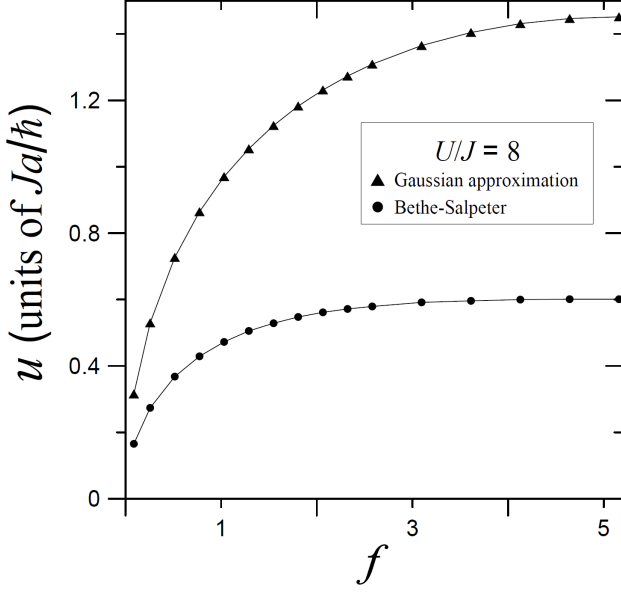


FIG. 2: The speed of sound in 2D optical lattice along the $(Q_x, 0)$ direction as a function of the filling factor f for s-wave pairing calculated within the Gaussian approximation, and by the BS formalism (solid lines are guides to the eyes). The interaction strength is $U = 8J$, and the filling factor has been normalized such that $f = 1$ at $\mu = 0$.

at $\mu = 0$. Using the s-wave pairing gap Δ_s , and the chemical potentials μ , presented in Fig. 1, we have calculated the speed of sound along the $(Q_x, 0)$ direction as a function of the filling factor f . The numerical results are shown in Fig. 2. As shown, the speed of sound is a smooth function across the BCS-BEC evolution. Also shown that the Gaussian approximation overestimates the speed of sound when compared to the BS formalism (or to the density-response-function approach).

One may well ask whether the BCS-BEC evolution is smooth crossover in fermion systems with d-wave pairing. Recently, the BCS-BEC evolution in a 2D fermion system with d-wave pairing has been studied in Ref. [5]. The d-wave pairing is naturally associated with near-neighbor interactions, but the assumption in Ref. [5] is that the short-coherence-length d-wave superfluidity is associated to the separable potential $V(\mathbf{k} - \mathbf{q}) = \lambda \Gamma(\mathbf{k}) \Gamma^*(\mathbf{q})$, where λ is the strength of the interaction, and $\Gamma(\mathbf{k} \rightarrow 0) \sim k^2$. Under this assumption, the collective-mode dispersion $\omega(\mathbf{Q})$ has been numerically calculated within the Gaussian-fluctuation approximation. This approximation provides a 2×2 secular determinant, and the speed of sound u (the slope of the linear part of the graph ω vs. Q) was obtained by replacing the exact elements of the secular determinant with their Taylor expansions about the points $Q = 0$ and $\omega = 0$. The result is that the speed of sound depends on a 2D-momentum integral, with this integral being divergent at the point $\mu = 0$ for d-wave pairing gap $\Delta_{\mathbf{k}} \sim \Gamma(\mathbf{k})$. The divergence of the integral is the reason to argue that the speed of sound is

non-analytic function across the BCS-BEC transition at $\mu = 0$, regardless of the specific details of the interaction potential.

In what follows, we report results concerning the speed of sound of a superfluid atomic spin balanced Fermi gas in a 2D optical lattice across the BCS-BEC evolution assuming $d_{x^2-y^2}$ -wave pairing gap of $\Delta_{\mathbf{k}} = \Delta (\cos k_x - \cos k_y) / 2$, where Δ is the maximum value of the gap. d-wave pairing gap has been observed in high- T_c -cuprates. The cuprate superconductors are close to an antiferromagnetic instability driven by a strong on-site repulsion U , and the near-neighbor interactions are the spin independent and attractive V , and the Heisenberg-type antiferromagnetic. In optical lattices, the $d_{x^2-y^2}$ -wave gap is associated with the attractive interaction between the atoms on the near-neighbor sites of the lattice. Decades ago, it was pointed out that the phase diagram at half filling of the extended Hubbard model shows an "island" in U - V space where d-wave pairing exists.⁶ There are no experimental data suggesting that a quantum phase transition at vanishing chemical potential takes place across the BCS-BEC evolution in systems described by the extended Hubbard model.

In this communication, we evaluate the collective-excitation energies of the extended Hubbard model in the generalized random phase approximation (GRPA). The Hamiltonian of the extended Hubbard model contains two interactions representing the on-site Hubbard interaction U and the attractive interaction V between the atoms on the near-neighbor sites of the lattice:

$$H = - \sum_{i,j,\sigma} J_{ij} \psi_{i,\sigma}^\dagger \psi_{j,\sigma} - \mu \sum_{i,\sigma} \hat{n}_{i,\sigma} - U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} - V \sum_{\langle i,j \rangle > \sigma \sigma'} \hat{n}_{i,\sigma} \hat{n}_{j,\sigma'}. \quad (5)$$

The Fermi operator $\psi_{i,\sigma}^\dagger$ ($\psi_{i,\sigma}$) creates (destroys) a fermion on the lattice site i with spin projection $\sigma = \uparrow, \downarrow$ along a specified direction, and $\hat{n}_{i,\sigma} = \psi_{i,\sigma}^\dagger \psi_{i,\sigma}$ is the density operator on site i . The symbol $\sum_{\langle ij \rangle}$ means sum over nearest-neighbor sites of a square lattice. The first term in (5) is the usual kinetic energy term in a tight-binding approximation, where J_{ij} is the tunneling strength of the atoms between sites i and j . The total number of sites is N , and the number of fermion atoms is M (filling factor $f = M/N$).

In the presence of the nearest-neighbor interaction, the gap equation (1) becomes

$$\Delta_{\mathbf{k}} = \frac{1}{N} \sum_{\mathbf{q}} [U + V(\mathbf{k} - \mathbf{q})] \frac{\Delta_{\mathbf{q}}}{2E(\mathbf{q})}, \quad (6)$$

where $E(\mathbf{k}) = \sqrt{\xi^2(\mathbf{k}) + \Delta_{\mathbf{k}}^2}$, and the interaction is $V(\mathbf{k}) = 2\lambda(\cos k_x + \cos k_y)$. Since the interaction can be factorized as $V(\mathbf{k} - \mathbf{q}) = \lambda[\Gamma_1(\mathbf{k})\Gamma_1(\mathbf{q}) + \Gamma_2(\mathbf{k})\Gamma_2(\mathbf{q}) + \Gamma_3(\mathbf{k})\Gamma_3(\mathbf{q}) + \Gamma_4(\mathbf{k})\Gamma_4(\mathbf{q})]$, where $\Gamma_1(\mathbf{k}) = \cos k_x - \cos k_y$, $\Gamma_2(\mathbf{k}) = \sin k_x - \sin k_y$, $\Gamma_3(\mathbf{k}) = \cos k_x + \cos k_y$, $\Gamma_4(\mathbf{k}) = \sin k_x + \sin k_y$, the gap is defined as

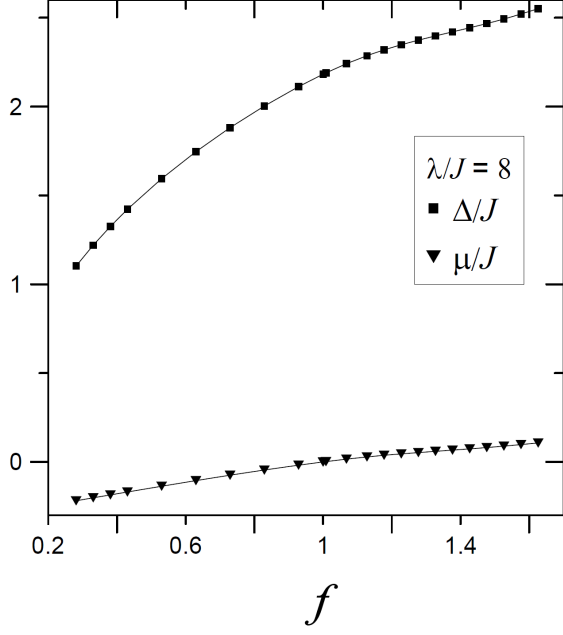


FIG. 3: The maximum value of the gap Δ , and the chemical potential μ as functions of filling factor f (solid lines are guides to the eyes). The interaction strength is $\lambda = 8J$, and the filling factor has been normalized such that $f = 1$ at $\mu = 0$.

$\Delta_{\mathbf{k}} = \Delta_s + \Delta\Gamma_1(\mathbf{k})/2 + \Delta_1\Gamma_3(\mathbf{k})/2$. If the Hubbard on-site interaction is weak, the s-wave pairing gap Δ_s can be neglected. The $d_{x^2-y^2}$ -wave pairing in the phase diagram appears in the limit $\Delta \gg \Delta_1$. In this limit, the chemical potential μ and the maximum value of the gap Δ are defined by the solutions of the mean-field gap- and number-equations:

$$1 = \frac{\lambda}{N} \sum_{\mathbf{k}} \frac{(\cos k_x - \cos k_y)^2}{2E(\mathbf{k})}, \quad (7)$$

$$f = \frac{1}{N} \sum_{\mathbf{k}} \left(1 - \frac{\xi(\mathbf{k})}{E(\mathbf{k})} \right). \quad (8)$$

The above mean-field number- and gap-equations were solved at zero temperature. The corresponding mean-field results for the maximum value of the gap Δ and the chemical potential μ (in units J) are shown in Fig. 3. In the numerical calculations, the strength of the nearest-neighbor interaction is fixed at $\lambda = 8J$. The chemical potential vanishes at a filling factor $f = 0.0502$.

The solutions of Eqs. (7) and (8) correspond to the minimum of the zero-temperature free energy of the system $F(\mu, \Delta) = \Delta^2/\lambda + 1/N \sum_{\mathbf{k}} [\xi(\mathbf{k}) - E(\mathbf{k})] + f\mu$. In Ref. [5], the derivative $\partial f/\partial \mu$ is divergent for $\mathbf{k} \rightarrow 0$ at $\mu = 0$. The point $\mathbf{k} = 0$ can be eliminated in all summations over the momentum vectors by rewriting the free energy as $F(\mu, \Delta) = \Delta^2/\lambda + 1/N \sum_{\mathbf{k}(\mathbf{k} \neq 0)} [\xi(\mathbf{k}) - E(\mathbf{k})] + \tilde{f}\mu$, where the effective filling factor is $\tilde{f} = f + \delta f$, and $\mu\delta f = [\xi(\mathbf{k}=0) - E(\mathbf{k}=0)]/N$. At a fixed filling factor \tilde{f} , the mean-field number- and gap-equations are: $\tilde{f} = \frac{1}{N} \sum_{\mathbf{k}(\mathbf{k} \neq 0)} \left(1 - \frac{\xi(\mathbf{k})}{E(\mathbf{k})} \right)$, and $1 = \frac{\lambda}{N} \sum_{\mathbf{k}(\mathbf{k} \neq 0)} \frac{(\cos k_x - \cos k_y)^2}{2E(\mathbf{k})}$.

II. BETHE-SALPETER EQUATIONS FOR THE COLLECTIVE MODES OF THE EXTENDED HUBBARD MODEL

The basic assumption in our BS formalism is that the bound states of two fermions in the optical lattice at zero temperature are described by the BS wave functions (BS amplitudes). The BS amplitude determines the probability of finding the first fermion at site \mathbf{r}_i with a spin σ_1 at the moment t_1 , and the second fermion at site \mathbf{r}_j with a spin σ_2 at the moment t_2 . The BS amplitudes depend on the relative internal time $t_1 - t_2$ and on the "center-of-mass" time $(t_1 + t_2)/2$: $\Phi_{\sigma_1, \sigma_2}^{\mathbf{Q}}(\mathbf{r}_i, \mathbf{r}_j; t_1, t_2) = e^{i[\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2 - \omega(\mathbf{Q})(t_1 + t_2)/2]} \Psi_{\sigma_1, \sigma_2}^{\mathbf{Q}}(\mathbf{r}_i - \mathbf{r}_j, t_1 - t_2)$, where \mathbf{Q} and $\omega(\mathbf{Q})$ are the collective-mode momentum and the corresponding dispersion, respectively. In the case of the extended Hubbard model, the kernel I of the BS equation $[K^{(0)-1} - I]\Psi = 0$ is time independent; therefore, one can use the BS equation for the equal-time BS amplitudes $\Psi_{\sigma_1, \sigma_2}^{\mathbf{Q}}(\mathbf{r}_i - \mathbf{r}_j, 0)$. The free two-particle propagator $K^{(0)}$ in the BS equation is a product of two fully dressed single-particle Green's functions, but in the GRPA the fully dressed single-particle Green's functions were replaced by the corresponding Green's functions $\begin{pmatrix} G^{\uparrow, \uparrow}(\mathbf{k}, \mathbf{Q}, \omega) & G^{\uparrow, \downarrow}(\mathbf{k}, \mathbf{Q}, \omega) \\ G^{\downarrow, \uparrow}(\mathbf{k}, \mathbf{Q}, \omega) & G^{\downarrow, \downarrow}(\mathbf{k}, \mathbf{Q}, \omega) \end{pmatrix}$ in the mean-field approximation. By introducing the functions $G^{\pm}(\mathbf{k}, \mathbf{Q})$, the Fourier transforms of $\Psi_{\sigma_1, \sigma_2}^{\mathbf{Q}}(\mathbf{r}_i - \mathbf{r}_j, 0)$ can be written as:

$$\begin{aligned}
\Psi^{\downarrow,\uparrow}(\mathbf{k}, \mathbf{Q}) &= [(l_{\mathbf{k},\mathbf{Q}} + \gamma_{\mathbf{k},\mathbf{Q}})G^+(\mathbf{k}, \mathbf{Q}) + (l_{\mathbf{k},\mathbf{Q}} - \gamma_{\mathbf{k},\mathbf{Q}})G^-(\mathbf{k}, \mathbf{Q})] / 2, \\
\Psi^{\uparrow,\downarrow}(\mathbf{k}, \mathbf{Q}) &= [(l_{\mathbf{k},\mathbf{Q}} - \gamma_{\mathbf{k},\mathbf{Q}})G^+(\mathbf{k}, \mathbf{Q}) + (l_{\mathbf{k},\mathbf{Q}} + \gamma_{\mathbf{k},\mathbf{Q}})G^-(\mathbf{k}, \mathbf{Q})] / 2, \\
\Psi^{\uparrow,\uparrow}(\mathbf{k}, \mathbf{Q}) &= [(\tilde{\gamma}_{\mathbf{k},\mathbf{Q}} - m_{\mathbf{k},\mathbf{Q}})G^+(\mathbf{k}, \mathbf{Q}) - (m_{\mathbf{k},\mathbf{Q}} + \tilde{\gamma}_{\mathbf{k},\mathbf{Q}})G^-(\mathbf{k}, \mathbf{Q})] / 2, \\
\Psi^{\downarrow,\downarrow}(\mathbf{k}, \mathbf{Q}) &= [(\tilde{\gamma}_{\mathbf{k},\mathbf{Q}} + m_{\mathbf{k},\mathbf{Q}})G^+(\mathbf{k}, \mathbf{Q}) + (m_{\mathbf{k},\mathbf{Q}} - \tilde{\gamma}_{\mathbf{k},\mathbf{Q}})G^-(\mathbf{k}, \mathbf{Q})] / 2.
\end{aligned}$$

Here the form factors are defined in the same manner as in Ref. [7]: $\gamma_{\mathbf{k},\mathbf{Q}} = u_{\mathbf{k}}u_{\mathbf{k}+\mathbf{Q}} + v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{Q}}$, $l_{\mathbf{k},\mathbf{Q}} = u_{\mathbf{k}}u_{\mathbf{k}+\mathbf{Q}} - v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{Q}}$, $\tilde{\gamma}_{\mathbf{k},\mathbf{Q}} = u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{Q}} - u_{\mathbf{k}+\mathbf{Q}}v_{\mathbf{k}}$, and $m_{\mathbf{k},\mathbf{Q}} = u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{Q}} + u_{\mathbf{k}+\mathbf{Q}}v_{\mathbf{k}}$, where $u_{\mathbf{k}} =$

$\sqrt{[1 + \xi_{\mathbf{k}}/E(\mathbf{k})]/2}$, and $v_{\mathbf{k}} = \sqrt{[1 - \xi_{\mathbf{k}}/E(\mathbf{k})]/2}$.

The functions $G^{\pm}(\mathbf{k}, \mathbf{Q})$, as well as the collective-excitation energy $\omega(\mathbf{Q})$, are defined by the solutions of the following coupled equations:⁸

$$\begin{aligned}
[\omega(\mathbf{Q}) - \varepsilon(\mathbf{k}, \mathbf{Q})]G^+(\mathbf{k}, \mathbf{Q}) &= -\frac{U}{2N} \sum_{\mathbf{q}} [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} + l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}}] G^+(\mathbf{q}, \mathbf{Q}) + \frac{U}{2N} \sum_{\mathbf{q}} [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} - l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}}] G^-(\mathbf{q}, \mathbf{Q}) \\
&- \frac{1}{2N} \sum_{\mathbf{q}} V(\mathbf{k} - \mathbf{q}) [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} + l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}} + \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} + m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}}] G^+(\mathbf{q}, \mathbf{Q}) \\
&+ \frac{1}{2N} \sum_{\mathbf{q}} V(\mathbf{k} - \mathbf{q}) [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} - l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}} + \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} - m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}}] G^-(\mathbf{q}, \mathbf{Q}) \\
&+ \frac{U}{2N} \sum_{\mathbf{q}} \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} [G^+(\mathbf{q}, \mathbf{Q}) - G^-(\mathbf{q}, \mathbf{Q})] - \frac{U + 2V(\mathbf{Q})}{2N} \sum_{\mathbf{q}} m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}} [G^+(\mathbf{q}, \mathbf{Q}) + G^-(\mathbf{q}, \mathbf{Q})],
\end{aligned} \tag{9}$$

$$\begin{aligned}
[\omega(\mathbf{Q}) + \varepsilon(\mathbf{k}, \mathbf{Q})]G^-(\mathbf{k}, \mathbf{Q}) &= \frac{U}{2N} \sum_{\mathbf{q}} [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} + l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}}] G^-(\mathbf{q}, \mathbf{Q}) - \frac{U}{2N} \sum_{\mathbf{q}} [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} - l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}}] G^+(\mathbf{q}, \mathbf{Q}) \\
&+ \frac{1}{2N} \sum_{\mathbf{q}} V(\mathbf{k} - \mathbf{q}) [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} + l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}} + \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} + m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}}] G^-(\mathbf{q}, \mathbf{Q}) \\
&- \frac{1}{2N} \sum_{\mathbf{q}} V(\mathbf{k} - \mathbf{q}) [\gamma_{\mathbf{k},\mathbf{Q}}\gamma_{\mathbf{q},\mathbf{Q}} - l_{\mathbf{k},\mathbf{Q}}l_{\mathbf{q},\mathbf{Q}} + \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} - m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}}] G^+(\mathbf{q}, \mathbf{Q}) \\
&+ \frac{U}{2N} \sum_{\mathbf{q}} \tilde{\gamma}_{\mathbf{k},\mathbf{Q}}\tilde{\gamma}_{\mathbf{q},\mathbf{Q}} [G^+(\mathbf{q}, \mathbf{Q}) - G^-(\mathbf{q}, \mathbf{Q})] + \frac{U + 2V(\mathbf{Q})}{2N} \sum_{\mathbf{q}} m_{\mathbf{k},\mathbf{Q}}m_{\mathbf{q},\mathbf{Q}} [G^+(\mathbf{q}, \mathbf{Q}) + G^-(\mathbf{q}, \mathbf{Q})],
\end{aligned} \tag{10}$$

where $\varepsilon(\mathbf{k}, \mathbf{Q}) = E(\mathbf{k} + \mathbf{Q}) + E(\mathbf{k})$.

The first four terms in the right-hand sites (RHS) of the BS equations represent the direct interactions in the BS kernel, while the last two terms in the RHS correspond to the exchange interactions. The existence of two U-dependent exchange terms is due to the fact that the Hubbard interaction has a spin-dependent part and a spin-independent part.⁹ The spin-independent part of the interaction appears in combination with the spin-independent interaction $V(\mathbf{Q})$ between the atoms on the near-neighbor sites of the lattice.

As it is known, the gauge invariance is restored by the existence of the Goldstone mode whose $\omega(\mathbf{Q})$ energy approaches zero at $\mathbf{Q} = 0$. From the BS point

of view, the Goldstone theorem corresponds to the so-called trivial solution of the BS equations: $G^+(\mathbf{k}, \mathbf{Q} = \mathbf{0}) = -G^-(\mathbf{k}, \mathbf{Q} = \mathbf{0}) = \Delta_{\mathbf{k}}/2E(\mathbf{k}) = \Psi^{\downarrow,\uparrow}(\mathbf{k}, \mathbf{Q} = \mathbf{0}) = -\Psi^{\uparrow,\downarrow}(\mathbf{k}, \mathbf{Q} = \mathbf{0})$, $\Psi^{\uparrow,\uparrow}(\mathbf{k}, \mathbf{Q} = \mathbf{0}) = \Psi^{\downarrow,\downarrow}(\mathbf{k}, \mathbf{Q} = \mathbf{0}) = 0$. As it is easy to see, the trivial solution recovers the gap equation (6).

By setting the near-neighbor interaction equal to zero, it is straightforward to prove that the existence of a nontrivial solution requires that the secular determinant (4) is equal to zero.

We now evaluate the collective-mode energy by setting the Hubbard interaction to zero. First, we set $\Gamma_2(\mathbf{k}) = \Gamma_3(\mathbf{k}) = \Gamma_4(\mathbf{k}) = 0$, and neglect $\tilde{\gamma}_{\mathbf{k},\mathbf{Q}}$ and $m_{\mathbf{k},\mathbf{Q}}$ terms. The existence of a nontrivial solution requires that the

following secular determinant

$$\begin{vmatrix} \lambda^{-1} + I_{\gamma,\gamma}^{1,1} & J_{\gamma,l}^{1,1} \\ J_{\gamma,l}^{1,1} & \lambda^{-1} + I_{l,l}^{1,1} \end{vmatrix} = 0 \quad (11)$$

is equal to zero; therefore, the corresponding equation for the collective modes is

$$1 + \lambda \left(I_{\gamma,\gamma}^{1,1} + I_{l,l}^{1,1} \right) + \lambda^2 \left[I_{\gamma,\gamma}^{1,1} I_{l,l}^{1,1} - \left(J_{\gamma,l}^{1,1} \right)^2 \right] = 0, \quad (12)$$

where the quantities $I_{a,b}^{i,j}$ and $J_{a,b}^{i,j}$ are defined as follow:

$$I_{a,b}^{i,j} = \frac{1}{N} \sum_{\mathbf{k}} a_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}} \frac{\Gamma_i(\mathbf{k}) \Gamma_j(\mathbf{k}) \varepsilon(\mathbf{k}, \mathbf{Q})}{\omega^2(\mathbf{Q}) - \varepsilon^2(\mathbf{k}, \mathbf{Q})},$$

$$J_{a,b}^{i,j} = \frac{1}{N} \sum_{\mathbf{k}} a_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}} \frac{\Gamma_i(\mathbf{k}) \Gamma_j(\mathbf{k}) \omega(\mathbf{Q})}{\omega^2(\mathbf{Q}) - \varepsilon^2(\mathbf{k}, \mathbf{Q})}.$$

Here $\Gamma_0(\mathbf{k}) = 1$, and $a_{\mathbf{k},\mathbf{Q}}$ and $b_{\mathbf{k},\mathbf{Q}}$ are one of the following form factors: $\gamma_{\mathbf{k},\mathbf{Q}}$, $l_{\mathbf{k},\mathbf{Q}}$, $\tilde{\gamma}_{\mathbf{k},\mathbf{Q}}$ and $m_{\mathbf{k},\mathbf{Q}}$.

In Ref. [5], the collective modes across the BCS-BEC transition for d-wave pairing are obtained in the Gaussian approximation using the following secular determinant:

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{vmatrix}, \quad M_{11} = 1 + (\lambda/2) \left(I_{\gamma,\gamma}^{1,1} + I_{l,l}^{1,1} - 2J_{\gamma,l}^{1,1} \right),$$

$$M_{22} = 1 + (\lambda/2) \left(I_{\gamma,\gamma}^{1,1} + I_{l,l}^{1,1} + 2J_{\gamma,l}^{1,1} \right), \quad M_{12} =$$

$(\lambda/2) \left(I_{l,l}^{1,1} - I_{\gamma,\gamma}^{1,1} \right)$. This determinant provides the same equation as the BS equation (12). From the BS point of view, the Gaussian approximation for the collective-mode dispersion and the BS formalism both provide the same dispersion, if the form factors $\tilde{\gamma}_{\mathbf{k},\mathbf{Q}}$ and $m_{\mathbf{k},\mathbf{Q}}$ are equal to zero. If all $\Gamma_i(\mathbf{k})$ terms in the nearest-neighbor interaction are taken into account, the collective-mode dispersion within the Gaussian approximation is defined by the 8×8 secular determinant $G_{8 \times 8}$ given in the Appendix.

The existence of a nontrivial solution of the BS equations (9) and (10) beyond the Gaussian approximation ($\tilde{\gamma}_{\mathbf{k},\mathbf{Q}} \neq 0$ and $m_{\mathbf{k},\mathbf{Q}} \neq 0$) leads to the following 17×17 secular determinant

$$Z_{17 \times 17} = \begin{vmatrix} D_{16 \times 16} & N_{1 \times 16}^T \\ N_{1 \times 16} & [2V(\mathbf{Q})]^{-1} + I_{m,m}^{0,0} \end{vmatrix}.$$

The direct interactions in the BS kernel form the 16×16 block $D_{16 \times 16} = \begin{vmatrix} G_{8 \times 8} & A_{8 \times 8} \\ A_{8 \times 8} & B_{8 \times 8} \end{vmatrix}$. The rest of the elements of $Z_{17 \times 17}$ are related to the existence of the exchange interaction in the BS kernel. The blocks $A_{8 \times 8}$ and $B_{8 \times 8}$ and $N_{1 \times 16}$ are defined in the Appendix, and T means transposed matrix operation.

In Fig.4, we have presented the numerical results of the speed of sound along the $(Q_x, 0)$ direction as a function of the filling factor f . The speed of sound in the Gaussian approximation is obtained using the equation $\det[D_{8 \times 8}] = 0$, while the 17×17 secular determinant $Z_{17 \times 17}$ is used in the BS formalism. As can be seen, the speed of sound across the BCS-BEC transition is a smooth function.

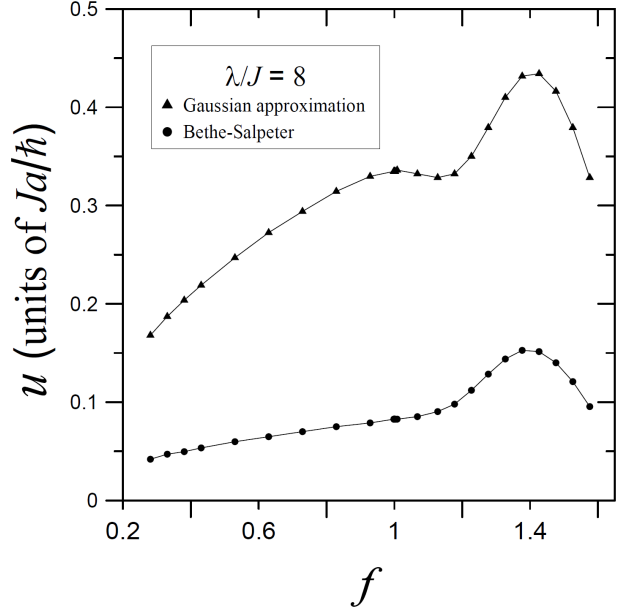


FIG. 4: The speed of sound in 2D optical lattice along the $(Q_x, 0)$ direction as a function of the filling factor f for d-wave pairing calculated within the Gaussian approximation, and by solving the BS equations (solid lines are guides to the eyes). The interaction strength is $\lambda = 8J$, and the filling factor has been normalized such that $f = 1$ at $\mu = 0$.

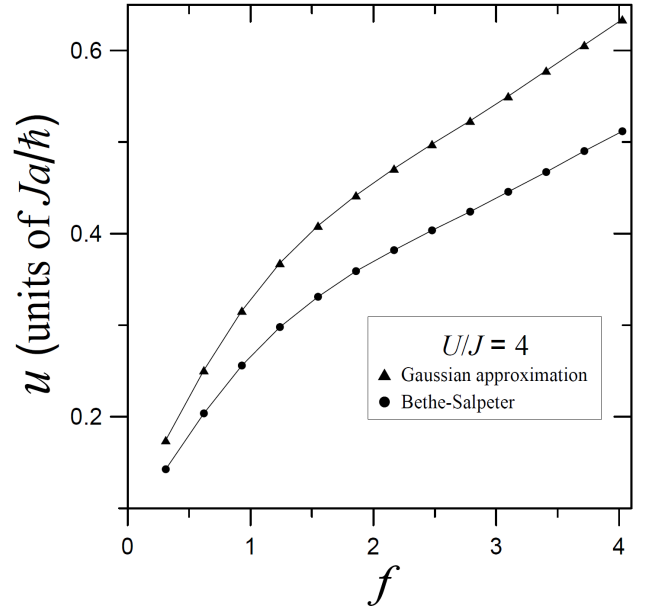


FIG. 5: The speed of sound in 2D optical lattice along the $(Q_x, 0)$ direction as a function of the filling factor f for s-wave pairing calculated within the Gaussian approximation, and by the BS formalism (solid lines are guides to the eyes). The interaction strength is $U = 4J$, and the filling factor has been normalized such that $f = 1$ at $\mu = 0$.

III. SUMMARY

To summarize, we argued that the infrared divergence predicted in Ref. [5] does not exist because in the BS formalism all elements of the secular determinant are defined only at points where $\mathbf{Q} \neq 0$. At the point $\mathbf{Q} = 0$ the coupled BS equations are replaced by a single equation (the gap equation); therefore, the secular determinant does not exist, and there is no infrared divergence.

It is worth mentioning another important point, illustrated in Fig. 2 and Fig. 4. The comparison between the BS and the Gaussian methods leads to the conclusion that, in the case of s- and d-wave pairings, the speed of sound calculated within the Gaussian approximation is overestimated. The use of the Gaussian approximation away from $f = 1$ have to be justified by comparing it with

the corresponding BS predictions. The results in Fig. 5 suggest that if the interaction decreases its strength, the difference between the Gaussian approach and the BS formalism becomes smaller.

IV. ACKNOWLEDGEMENTS

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V. APPENDIX

In the case of d-wave pairing ($U = 0$ and $\lambda > 0$), the secular determinant in the Gaussian approximation is

$$G_{8 \times 8} = \begin{vmatrix} \lambda^{-1} + I_{\gamma,\gamma}^{1,1} & I_{\gamma,\gamma}^{1,2} & I_{\gamma,\gamma}^{1,3} & I_{\gamma,\gamma}^{1,4} & J_{\gamma,l}^{1,1} & J_{\gamma,l}^{1,2} & J_{\gamma,l}^{1,3} & J_{\gamma,l}^{1,4} \\ I_{\gamma,\gamma}^{1,2} & \lambda^{-1} + I_{\gamma,\gamma}^{2,2} & I_{\gamma,\gamma}^{2,3} & I_{\gamma,\gamma}^{2,4} & J_{\gamma,l}^{1,2} & J_{\gamma,l}^{2,2} & J_{\gamma,l}^{2,3} & J_{\gamma,l}^{2,4} \\ I_{\gamma,\gamma}^{1,3} & I_{\gamma,\gamma}^{2,3} & \lambda^{-1} + I_{\gamma,\gamma}^{3,3} & I_{\gamma,\gamma}^{3,4} & J_{\gamma,l}^{1,3} & J_{\gamma,l}^{2,3} & J_{\gamma,l}^{3,3} & J_{\gamma,l}^{3,4} \\ I_{\gamma,\gamma}^{1,4} & I_{\gamma,\gamma}^{2,4} & I_{\gamma,\gamma}^{3,4} & \lambda^{-1} + I_{\gamma,\gamma}^{4,4} & J_{\gamma,l}^{1,4} & J_{\gamma,l}^{2,4} & J_{\gamma,l}^{3,4} & J_{\gamma,l}^{4,4} \\ J_{\gamma,l}^{1,1} & J_{\gamma,l}^{1,2} & J_{\gamma,l}^{1,3} & J_{\gamma,l}^{1,4} & \lambda^{-1} + I_{l,l}^{1,1} & I_{l,l}^{1,2} & I_{l,l}^{1,3} & I_{l,l}^{1,4} \\ J_{\gamma,l}^{1,2} & J_{\gamma,l}^{2,2} & J_{\gamma,l}^{2,3} & J_{\gamma,l}^{2,4} & I_{l,l}^{1,2} & \lambda^{-1} + I_{l,l}^{2,2} & I_{l,l}^{2,3} & I_{l,l}^{2,4} \\ J_{\gamma,l}^{1,3} & J_{\gamma,l}^{2,3} & J_{\gamma,l}^{3,3} & J_{\gamma,l}^{3,4} & I_{l,l}^{1,3} & I_{l,l}^{2,3} & \lambda^{-1} + I_{l,l}^{3,3} & I_{l,l}^{3,4} \\ J_{\gamma,l}^{1,4} & J_{\gamma,l}^{2,4} & J_{\gamma,l}^{3,4} & J_{\gamma,l}^{4,4} & I_{l,l}^{1,4} & I_{l,l}^{2,4} & I_{l,l}^{3,4} & \lambda^{-1} + I_{l,l}^{4,4} \end{vmatrix}. \quad (13)$$

Beyond the Gaussian approximation, the following additional blocks appear in the secular determinant $Z_{17 \times 17}$:

$$A_{8 \times 8} = \begin{vmatrix} I_{\gamma,\tilde{\gamma}}^{1,1} & I_{\gamma,\tilde{\gamma}}^{1,2} & I_{\gamma,\tilde{\gamma}}^{1,3} & I_{\gamma,\tilde{\gamma}}^{1,4} & J_{l,\tilde{\gamma}}^{1,1} & J_{l,\tilde{\gamma}}^{1,2} & J_{l,\tilde{\gamma}}^{1,3} & J_{l,\tilde{\gamma}}^{1,4} \\ I_{\gamma,\tilde{\gamma}}^{1,2} & I_{\gamma,\tilde{\gamma}}^{2,2} & I_{\gamma,\tilde{\gamma}}^{2,3} & I_{\gamma,\tilde{\gamma}}^{2,4} & J_{l,\tilde{\gamma}}^{1,2} & J_{l,\tilde{\gamma}}^{2,2} & J_{l,\tilde{\gamma}}^{2,3} & J_{l,\tilde{\gamma}}^{2,4} \\ I_{\gamma,\tilde{\gamma}}^{1,3} & I_{\gamma,\tilde{\gamma}}^{2,3} & I_{\gamma,\tilde{\gamma}}^{3,3} & I_{\gamma,\tilde{\gamma}}^{3,4} & J_{l,\tilde{\gamma}}^{1,3} & J_{l,\tilde{\gamma}}^{2,3} & J_{l,\tilde{\gamma}}^{3,3} & J_{l,\tilde{\gamma}}^{3,4} \\ I_{\gamma,\tilde{\gamma}}^{1,4} & I_{\gamma,\tilde{\gamma}}^{2,4} & I_{\gamma,\tilde{\gamma}}^{3,4} & I_{\gamma,\tilde{\gamma}}^{4,4} & J_{l,\tilde{\gamma}}^{1,4} & J_{l,\tilde{\gamma}}^{2,4} & J_{l,\tilde{\gamma}}^{3,4} & J_{l,\tilde{\gamma}}^{4,4} \\ I_{l,\tilde{\gamma}}^{1,1} & I_{l,\tilde{\gamma}}^{1,2} & I_{l,\tilde{\gamma}}^{1,3} & I_{l,\tilde{\gamma}}^{1,4} & \lambda^{-1} + I_{m,m}^{1,1} & I_{m,m}^{1,2} & I_{m,m}^{1,3} & I_{m,m}^{1,4} \\ I_{l,\tilde{\gamma}}^{1,2} & I_{l,\tilde{\gamma}}^{2,2} & I_{l,\tilde{\gamma}}^{2,3} & I_{l,\tilde{\gamma}}^{2,4} & I_{m,m}^{1,2} & \lambda^{-1} + I_{m,m}^{2,2} & I_{m,m}^{2,3} & I_{m,m}^{2,4} \\ I_{l,\tilde{\gamma}}^{1,3} & I_{l,\tilde{\gamma}}^{2,3} & I_{l,\tilde{\gamma}}^{3,3} & I_{l,\tilde{\gamma}}^{3,4} & I_{m,m}^{1,3} & I_{m,m}^{2,3} & \lambda^{-1} + I_{m,m}^{3,3} & I_{m,m}^{3,4} \\ I_{l,\tilde{\gamma}}^{1,4} & I_{l,\tilde{\gamma}}^{2,4} & I_{l,\tilde{\gamma}}^{3,4} & I_{l,\tilde{\gamma}}^{4,4} & I_{m,m}^{1,4} & I_{m,m}^{2,4} & I_{m,m}^{3,4} & \lambda^{-1} + I_{m,m}^{4,4} \end{vmatrix}, \quad (14)$$

$$B_{8 \times 8} = \begin{vmatrix} \lambda^{-1} + I_{\gamma,\tilde{\gamma}}^{1,1} & I_{\gamma,\tilde{\gamma}}^{1,2} & I_{\gamma,\tilde{\gamma}}^{1,3} & I_{\gamma,\tilde{\gamma}}^{1,4} & J_{\gamma,m}^{1,1} & J_{\gamma,m}^{1,2} & J_{\gamma,m}^{1,3} & J_{\gamma,m}^{1,4} \\ I_{\gamma,\tilde{\gamma}}^{1,2} & \lambda^{-1} + I_{\gamma,\tilde{\gamma}}^{2,2} & I_{\gamma,\tilde{\gamma}}^{2,3} & I_{\gamma,\tilde{\gamma}}^{2,4} & J_{\gamma,m}^{1,2} & J_{\gamma,m}^{2,2} & J_{\gamma,m}^{2,3} & J_{\gamma,m}^{2,4} \\ I_{\gamma,\tilde{\gamma}}^{1,3} & I_{\gamma,\tilde{\gamma}}^{2,3} & \lambda^{-1} + I_{\gamma,\tilde{\gamma}}^{3,3} & I_{\gamma,\tilde{\gamma}}^{3,4} & J_{\gamma,m}^{1,3} & J_{\gamma,m}^{2,3} & J_{\gamma,m}^{3,3} & J_{\gamma,m}^{3,4} \\ I_{\gamma,\tilde{\gamma}}^{1,4} & I_{\gamma,\tilde{\gamma}}^{2,4} & I_{\gamma,\tilde{\gamma}}^{3,4} & \lambda^{-1} + I_{\gamma,\tilde{\gamma}}^{4,4} & J_{\gamma,m}^{1,4} & J_{\gamma,m}^{2,4} & J_{\gamma,m}^{3,4} & J_{\gamma,m}^{4,4} \\ J_{\gamma,m}^{1,1} & J_{\gamma,m}^{1,2} & J_{\gamma,m}^{1,3} & J_{\gamma,m}^{1,4} & \lambda^{-1} + I_{m,m}^{1,1} & I_{m,m}^{1,2} & I_{m,m}^{1,3} & I_{m,m}^{1,4} \\ J_{\gamma,m}^{1,2} & J_{\gamma,m}^{2,2} & J_{\gamma,m}^{2,3} & J_{\gamma,m}^{2,4} & I_{m,m}^{1,2} & \lambda^{-1} + I_{m,m}^{2,2} & I_{m,m}^{2,3} & I_{m,m}^{2,4} \\ J_{\gamma,m}^{1,3} & J_{\gamma,m}^{2,3} & J_{\gamma,m}^{3,3} & J_{\gamma,m}^{3,4} & I_{m,m}^{1,3} & I_{m,m}^{2,3} & \lambda^{-1} + I_{m,m}^{3,3} & I_{m,m}^{3,4} \\ J_{\gamma,m}^{1,4} & J_{\gamma,m}^{2,4} & J_{\gamma,m}^{3,4} & J_{\gamma,m}^{4,4} & I_{m,m}^{1,4} & I_{m,m}^{2,4} & I_{m,m}^{3,4} & \lambda^{-1} + I_{m,m}^{4,4} \end{vmatrix}, \quad (15)$$

$$N_{1 \times 16} = \left(J_{\gamma,m}^{1,0}, J_{\gamma,m}^{2,0}, J_{\gamma,m}^{3,0}, J_{\gamma,m}^{4,0}, I_{l,m}^{1,0}, I_{l,m}^{2,0}, I_{l,m}^{3,0}, I_{l,m}^{4,0}, J_{\gamma,m}^{1,0}, J_{\gamma,m}^{2,0}, J_{\gamma,m}^{3,0}, J_{\gamma,m}^{4,0}, I_{m,m}^{1,0}, I_{m,m}^{2,0}, I_{m,m}^{3,0}, I_{m,m}^{4,0} \right). \quad (16)$$

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